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THE DYNAMIC CONTACT PROBLEM FOR A CIRCULAR PUNCH ADHERING TO AN ELASTIC LAYER†

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As an extension of the approach proposed in [1] for the problem of smooth contact, a method of solving dynamic contact problems is described for the case of a circular punch which adheres to an elastic layer. An infinite system of linear algebraic equations is obtained which, taking into account the asymptotic behaviour of the unknowns, reduces to an asymptotically equivalent finite system. The model yields simple relations between the unknowns corresponding to the tangential and normal components, and the dimensions of the system is thereby halved. The work involved in the realization of the method is practically the same as in the case of smooth contact.

The results are given for punches adhering to an elastic base in cases where allowance must be made for the adhesive forces, as in the analysis of the stress–strained state in bonded and welded systems, or the dynamics of bases and foundations which are rigidly embedded in the soil. Examples of elastic bases that have been considered are a uniform half-space [2–4] or layer [5–7], with circular [2, 4], strip [5, 6] or rectangular [3, 7] contact domains.

Apart from applications such as those mentioned, contact problems with friction need to be solved in order to determine the limits of applicability of the results obtained under weaker contact conditions. The results for the frequency range $0 \leqslant \omega a/v_s \leqslant 3$ (where a is the punch and radius v_s is the wave velocity of S-waves in the medium) for the case of a circular punch and a uniform half-space for $\nu = 0.25$ (ν is Poisson's ratio) were compared in [2].

The results of a similar comparison for an elastic layer with other values of Poisson's ratio are discussed.

1. WE CONSIDER steady harmonic vibrations of a punch of circular cross-section, the surface of which adheres to an elastic layer, rigidly bonded to a non-deformable foundation. A vertical axisymmetric load $Fe^{-i\omega t}$ is applied to the punch. In view of the linearity of the problem, the factor $e^{-i\omega t}$ will be ignored below.

To find the vertical displacement of the punch w, the wave fields and stream lines of energy, we first need to solve the contact problem for the unknown contact stresses

$$\mathbf{q} = \{\tau_{rz}, \sigma_{zz}\} = \{q_1, q_2\} \qquad (\tau_{\varphi z} \equiv 0)$$

By the use of a Fourier transformation with respect to the horizontal coordinates x, y, which in cylindrical coordinates r, φ, z ($z = 0$ is the surface of the layer) takes the form of a Fourier–Bessel transformation, the system of integral equations for $\mathbf{q}(r)$ for the case of equal displacements of the punch and surface of the layer in the contact zone reduces to the functional equation

(1.1)

Here

$$K(\alpha) = \begin{Vmatrix} -iM & -i\alpha P \\ i\alpha P & R \end{Vmatrix}, \quad M = \frac{M_1}{\Delta}, \quad P = \frac{P_1}{\Delta}, \quad R = \frac{R_1}{\Delta}$$

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$$\mathbf{Q}(\alpha) = 2\pi \int_0^\infty \left\| \begin{matrix} iJ_1(\alpha r) q_1(r) \\ J_0(\alpha r) q_2(r) \end{matrix} \right\| r dr \tag{1.2}$$

$$\mathbf{F} + \Phi = 2\pi \left(\int_0^a + \int_a^\infty \right) \left\| \begin{matrix} iJ_1(\alpha r) u_1(r) \\ J_0(\alpha r) u_2(r) \end{matrix} \right\| r dr$$

$$\mathbf{u} = \{ u_r, u_z \} = \{ u_1, u_2 \}$$

(u is the displacement of the surface of the layer).

From the condition $\mathbf{u}|_{r \le a} = \{0, w\}$ it follows that

$$\mathbf{F}(\alpha) = w\mathbf{F}_1(\alpha), \quad \mathbf{F}_1(\alpha) = \{ 0, 2\pi a J_1(a\alpha)/\alpha \}$$

For the given layer of unit thickness

$$M_1(\alpha) = i\sigma_2(\gamma - \alpha^2)[\alpha^2 d_{12} - \sigma_1 \sigma_2 d_{21}]/2$$

$$P_1(\alpha) = \sigma_1 \sigma_2 (\gamma + \alpha^2) + (\gamma \alpha^2 + \sigma_1^2 \sigma_2^2) s_{12} - \sigma_1 \sigma_2 (\gamma + \alpha^2) c_{12}$$

$$R_1(\alpha) = \sigma_1(\gamma - \alpha^2)[\alpha^2 d_{21} - \sigma_1 \sigma_2 d_{12}]$$

$$\Delta(\alpha) = (2\alpha^2 \sigma_1 \sigma_2 \gamma + \alpha^2 [\gamma^2 + \sigma_1^2 \sigma_2^2] s_{12} - [\gamma^2 + \alpha^4] \sigma_1 \sigma_2 c_{12}) \mu$$

$$d_{ij} = sh\, \sigma_i\, ch\, \sigma_j, \quad s_{12} = sh\, \sigma_1\, sh\, \sigma_2, \quad c_{12} = ch\, \sigma_1\, ch\, \sigma_2$$

$$\sigma_n = \sqrt{\alpha^2 - \kappa_n^2}, \quad n = 1, 2; \quad \gamma = \alpha^2 - \frac{\kappa_2^2}{2}, \quad \kappa_1 = \frac{\omega}{v_p}, \quad \kappa_2 = \frac{\omega}{v_s}$$

(v_p, v_s are the velocities of P- and S-waves in the layer).

The general method used to reduce the problem to an infinite system is the same as that developed for smooth contact in [1]. In the case of adhesive contact considered here we have

$$\Phi(\alpha) = 2\pi a \sum_{k=1}^{\infty} \left\| \begin{matrix} it_{1,k} B_{1,k}(\alpha) \\ t_{2,k} B_{0,k}(\alpha) \end{matrix} \right\| (\alpha^2 - \zeta_k^2)^{-1}$$

$$B_{1,k}(\alpha) = \alpha J_0(a\alpha) H_1^{(1)}(\zeta_k a) - \zeta_k J_1(a\alpha) H_0^{(1)}(a\zeta_k)$$

$$B_{0,k}(\alpha) = \zeta_k J_0(a\alpha) H_1^{(1)}(\zeta_k a) - \alpha J_1(a\alpha) H_0^{(1)}(a\zeta_k)$$

$$2t_{1,k} = -i\zeta_k (M_{1,k} Q_{1,k} + \zeta_k P_{1,k} Q_{3,k}) / \Delta'(\zeta_k)$$

$$2t_{2,k} = i\zeta_k (i\zeta_k P_{1,k} Q_{1,k} + R_{1,k} Q_{3,k}) / \Delta'(\zeta_k)$$

$$M_{1,k} = M_1(\zeta_k), \quad P_{1,k} = P_1(\zeta_k) \quad \text{и т.д.}$$

ζ_k are the poles of the elements $K(\alpha)$ [zeros $\Delta(\alpha)$] located in the complex plane α above the contour σ which circumvents real ζ_k as dictated by the limiting absorption principle [8].

From (1.1) if follows that

$$\mathbf{Q}(\zeta) = K^{-1}(\mathbf{F} + \Phi) = L(\zeta)(\mathbf{F}(\zeta) + \Phi(\zeta))/\Delta_2(\zeta)$$

$$L = \left\| \begin{matrix} -iR_1 & \alpha P_1 \\ -\alpha P_1 & -M_1 \end{matrix} \right\|, \quad \Delta_2 = 2\alpha^2 \sigma_1 \sigma_2 (1 - c_{12}) + (\alpha^4 + \sigma_1^2 \sigma_2^2) s_{12}$$

The function $\mathbf{Q}(\alpha)$, which is integral by construction, has no poles and thus the conditions for eliminating singularities, namely,

$$L(\zeta_k) \operatorname{res}(\mathbf{F}(\alpha) + \Phi(\alpha)) \,|_{\alpha = \zeta_k} = 0 \tag{1.3}$$

$$L(z_l)(F(z_l) + \Phi(z_l)) = 0; \quad l, k = 1, 2, \ldots \tag{1.4}$$

must be satisfied.

Here z_l are the zeros $\Delta_2(\alpha)$, also, to be specific, taken above the contour σ. Furthermore, the similar conditions for $\alpha = -\zeta_k$ and $\alpha = -z_l$ should be satisfied, but by virtue of their evenness, they reduce to (1.3) and (1.4).

From the relation $\det L = -i\Delta\Delta_2$ it follows that the rows of the matrices $L(\zeta_k)$ and $L(z_l)$ are linearly independent and for each l and k conditions (1.3) and (1.4) give one independent equation for the unknowns $t_{1,k}$ and $t_{2,k}$ which occur in $\Phi(\alpha)$.

Equations (1.3) reduce to the form

$$R_{1,k}t_{1,k} + \zeta_k P_{1,k}t_{2,k} = 0, \quad k = 1,2,\ldots \tag{1.5}$$

and, as in the case of smooth contact, this enables us to restrict our consideration to one set of unknowns $\mathbf{t} = \{t_{2,1}, t_{2,2}, \ldots\}$, by expressing $t_{1,k}$ in terms of $t_{2,k}$. The unknowns $\mathbf{t}$ are determined from the infinite system generated by condition (1.4)

$$A\mathbf{s} = \mathbf{f}, \quad A = \| a_{lk} \|_{l,k=1}^{\infty}, \quad \mathbf{s} = \{s_1, s_2, \ldots\}, \quad \mathbf{f} = \{f_1, f_2, \ldots\} \tag{1.6}$$

$$a_{lk} = \frac{1}{z_l^2 - \zeta_k^2}[-B_{1,k}(z_l)\frac{c_k}{d_l} + B_{0,k}(z_l)]\,(J_1(az_l)H_1^{(1)}(a\zeta_k))^{-1}$$

$$f_l = -\frac{1}{z_l}, \quad t_{2,k} = \frac{s_k}{H_1^{(1)}(a\zeta_k)}, \quad c_k = \frac{\zeta_k P_{1,k}}{R_{1,k}}, \quad d_l = \frac{z_l P_1(z_l)}{R_1(z_l)}$$

For smooth contact [1], the system was regularized by separating and inverting its principal part. The drawbacks of this method are its slow convergence in the neighbourhood of points where the real branches of the $z_l(\omega)$ curves emerge, and the large amount of computer time needed for matrix inversion and multiplication. The approach in which allowance is made for the asymptotic behaviour of the unknowns $t_{2,k}$ as $k \to \infty$ does not suffer from these drawbacks.

From the known behaviour of contact stresses in the neighbourhood of the punch boundary

$$\mathbf{q}(r) \sim \mathbf{q}_0^+(a-r)^{-1/2+i\gamma} + q_0^-(a-r)^{-1/2-i\gamma}, \quad r \to a$$

$$\mathbf{q}_0^{\pm} = \{q_{0,1}^{\pm}, q_{0,2}^{\pm}\}, \quad \gamma = \operatorname{arcth}\frac{1-2\nu}{2(1-\nu)}$$

and starting from the integral representation (1.2), we can write the asymptotic form of $Q(\alpha)$ as $|\alpha| \to \infty$ and, therefore, the asymptotic form of the unknowns as $k \to \infty$

$$t_{2,k} \sim 1/2i\zeta_k(N_{1,k}^+ q_{0,1}^+ + N_{2,k}^+ q_{0,2}^+ + N_{1,k}^- q_{0,1}^- + N_{2,k}^- q_{0,2}^-)/\Delta'(\zeta_k) = p_k \tag{1.7}$$

$$N_{1,k}^{\pm} = -\pi a\zeta_k P_1(\zeta_k)(2a/\zeta_k)^{1/2\pm i\gamma}\Gamma(1/2 \pm i\gamma)J_{3/2\pm i\gamma}(a\zeta_k)$$

$$N_{2,k}^{\pm} = \pi R_1(\zeta_k)(2a/\zeta_k)^{1/2\pm i\gamma}\Gamma(1/2 \pm i\gamma)J_{1/2\pm i\gamma}(a\zeta_k), \quad k \to \infty$$

Starting from a certain number $k = N+1$, if the asymptotic representation (1.7) is used for the unknowns $t_{2,k}$ in (1.4), we obtain an asymptotically equivalent finite system of dimension $N+4$, the solution of which becomes stable as N increases.

The use of the asymptotic form (1.7) enables us to determine not only the contact stresses $\mathbf{q}$ and contact stiffness $P_1 = w^{-1}/\iint_\Omega q_2 d\Omega$, but also the stress intensity factors $\mathbf{q}_0^1$ near the edge of the punch

$$\mathbf{q}(r) = \pi ia \sum_{l=1}^{\infty} \left\| \begin{matrix} -iJ_1(z_l r)[-iR_l S_1(z_l) + z_l P_l S_2(z_l)) \\ \\ J_0(z_l r)[-z_l P_l S_1(z_l) - M_l S_2(z_l)] \end{matrix} \right\| z_l - \left\| \begin{matrix} 0 \\ 2\dfrac{M_1(0)}{\Delta_2(0)} \end{matrix} \right\|$$

$$P_1 = \frac{\Delta(0)}{R_1(0)}[\pi a^2 - 2\pi a(\sum_{k=1}^{N}\frac{s_k}{\zeta_k} +$$

$$+ \sum_{k=N+1}^{\infty}\frac{i}{2\Delta'(\zeta_k)}(N_{1,k}^+ q_{0,1}^+ + N_{2,k}^+ q_{0,2}^+ + N_{1,k}^- q_{0,1}^- + N_{2,k}^- q_{0,2}^-))]$$

$$q_0^+ = \{t_{2,N+1}, t_{2,N+3}\}, \quad q_0^- = \{t_{2,N+2}, t_{2,N+1}\}$$

Here

$$R_l = \frac{R_1(z_l)}{\Delta_2'(z_l)}, \quad P_l = \frac{P_1(z_l)}{\Delta_2'(z_l)}, \quad M_l = \frac{M_1(z_l)}{\Delta_2'(z_l)}$$

$$S_1(z_l) = \sum_{k=1}^{N} \frac{1}{z_l^2 - \zeta_k^2}(-ic_k B_{1,k}^*(z_l) t_{2,k}) +$$

$$+ \sum_{k=N+1}^{\infty} \frac{1}{z_l^2 - \zeta_k^2}(-ic_k B_{1,k}^*(z_l)) p_k$$

$$S_2(z_l) = \frac{H_1^{(1)}(az_l)}{z_l} + \sum_{k=1}^{N} \frac{1}{z_l^2 - \zeta_k^2} B_{0,k}^*(z_l) t_{2,k} +$$

$$+ \sum_{k=N+1}^{\infty} \frac{1}{z_l^2 - \zeta_k^2} B_{0,k}^*(z_l) p_k$$

$$B_{1,k}^*(z_l) = [z_l H_0^{(1)}(z_l a) H_1^{(1)}(\zeta_k a) - \zeta_k H_1^{(1)}(z_l a) H_0^{(1)}(\zeta_k a)]$$

$$B_{0,k}^*(z_l) = [\zeta_k H_0^{(1)}(z_l a) H_1^{(1)}(\zeta_k a) - z_l H_1^{(1)}(z_l a) H_0^{(1)}(\zeta_k a)]$$

This method can also be used in the case of non-axisymmetric vibrations of a punch or deformable surface object. If the displacements in the contact region depend on the angular coordinate, instead of the functional Eq. (1.1), we obtain a denumerable set of equations which reduce to systems of the form (1.6), with the same matrix A, but different right-hand sides.

2. To examine the extent to which adhesion in the contact zone affects the dynamic contact stiffness P_1, a comparison was made with results obtained for smooth contact with $\nu = 0.1$; 0.3; 0.499997 and $a \in [0, 5]$ (where, as in previous papers, the thickness of the layer h, the velocity of S-waves v_s, and the density ρ are taken to be unity, that is, the units of all the dimensional quantities are expressed in terms of these three characteristic units). It was established that, for $0 \leq \omega \leq 3$, the results are practically the same for all ν, when $\omega > 3$ there is agreement for small a, but discrepancies arise as a increases, and for $\nu = 0.1$ there is little difference between the results for almost all the values of ω and a considered.

These features are illustrated by the graphs of the amplitude P_1 against ω in Figs 1–3.†

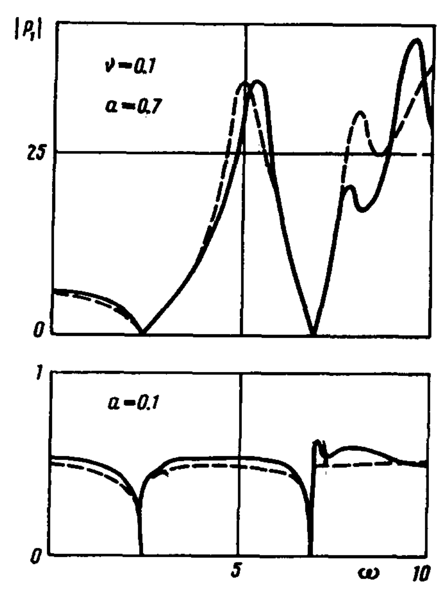

FIG. 1.

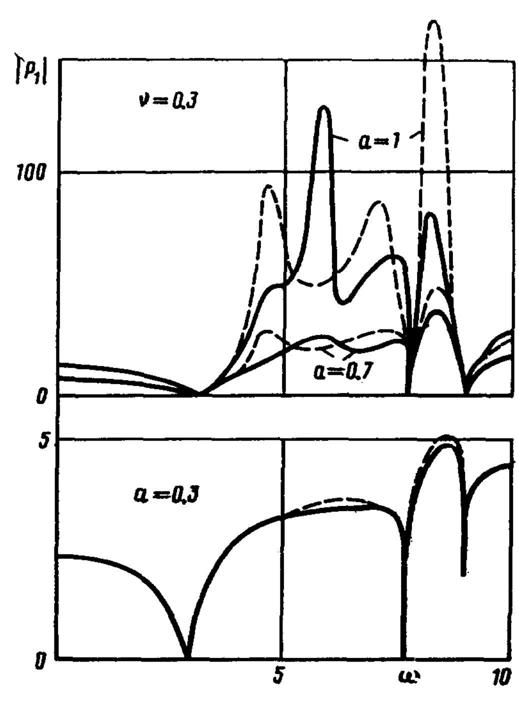

FIG. 2.

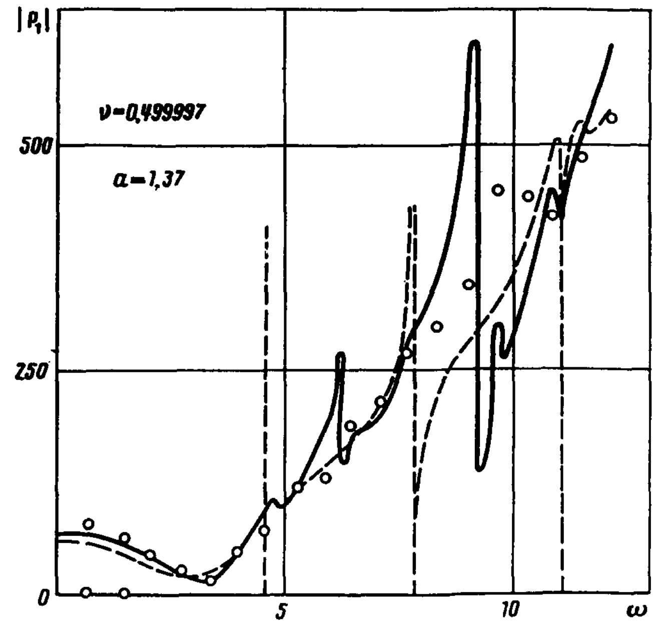

FIG. 3.

The results for smooth contact are represented by the dashed line; whenever the results are the same, to within the accuracy of the diagram, only the solid curve for contact with adhesion is shown. The open points in Fig. 3 represent the experimental results obtained by Ye. M. Timanin for a gelatine layer ($\nu = 0.499997$) which models the elastic properties of living tissue.

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† A more detailed set of graphs for all the cases examined, including the real and complex branches of the dispersion curves [the relations $z_l(\omega)$ and $\zeta_k(\omega)$] can be found in GLUSHKOV Ye. V., GLUSHKOVA N. V. and KIRILLOVA Ye. V., Calculation of the dynamic contact stiffness of an elastic layer in the case of adhesive contact. Unpublished paper, Kuban. Gos. Univ. Krasnodar. Deposited at VINITI 20.12.01, No. 4726-V91.